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Generic scaling relation in the scalar ϕ^4 model

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Abstract. The results of an analysis of the one-loop spectrum of the anomalous dimensions of composite operators in the scalar ϕ^4 model are presented. We give a rigorous constructive proof of the hypothesis on the hierarchical structure of the spectrum of anomalous dimensions—the naive sum of any two anomalous dimensions generates a limit point in the spectrum. Arguments in favour of the non-perturbative character of this result and the possible ways of generalizing it to other field theories are briefly discussed.

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1. Introduction

The theory of critical phenomena is one branch of physics where renormalization group (RG) methods have been very fruitful. It is now well established that the system at the phase transition point possesses a number of remarkable properties. The most important of these are the universality of the critical behaviour of physically different systems [1] and scale and even conformal invariance of the correlation functions [2]. The effectiveness of RG methods in the description of critical phenomena (the calculation of correlation functions, critical exponents, etc) is mainly explained by these properties [3].

However, the most impressive results had been obtained in two-dimensional conformal field theories. It was realized [4] that the restrictions posed by conformal invariance in the 2D case are highly non-trivial and lead (in principle) to the full description of the spectrum of the critical exponents. It would also be very interesting to understand which algebraic structure can be found in d-dimensional conformal field theories. However, beyond two dimensions conformal symmetry is well known to yield less stringent conditions. Therefore many problems in d-dimensional ($d > 2$) conformal field theory still remain unresolved.

In the recent papers [6, 7, 8, 9], analysis of the spectrum of the critical exponents in the $O(N)$-vector model in $4-\epsilon$ dimensions has been carried out in the framework of one-loop order perturbation theory. Due to relative simplicity of this model it appeared possible to obtain the exact solution of the eigenvalue problem for some classes of composite operators (see [6, 8]). In particular, analytical solutions have been obtained for the spectrum of the critical exponents of symmetric and traceless operators with number of fields $n \leqslant 4$ and arbitrary number of derivativites l. All these solutions reveal the distinctive regularity in the behaviour of the critical dimensions at large l. Moreover, the numerical analysis carried out in [7, 9] showed that the same regularity—the sum of two points of the spectrum of

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the critical exponents being the limit point of the latter—also holds for a wider class of operators.

In the present paper we give a rigorous proof of this property. In what follows we will restrict ourselves to the scalar ϕ^4 theory and consider the spatially symmetric and traceless operators only. However, all results will be valid for the class of $O(N)$ and spatially symmetric and traceless operators in the N-vector model [7].

Before proceeding with calculations we would like to discuss the main troubles which arise in the analysis of the spectrum of anomalous dimensions (where anomalous dim. = full scaling dim. − canonical dim.) of large spin operators. It is easy to understand that the source of all difficulties is the mixing problem. Indeed, a long time ago Callan and Gross proved a very strong statement concerning the anomalous dimensions of the twist-2 (where (twist = dimension − spin) operators [10] for which the mixing problem is absent. They obtained the result that for all orders of the perturbation theory the anomalous dimension λ_l of the operator $\phi\partial_{\mu_1}\cdots\partial_{\mu_l}\phi$ tends to $2\lambda_\phi$ at $l\to\infty$ (where λ_ϕ is the anomalous dimension of the field ϕ).

Let us see what prevents the generalization of this result, even on the one-loop level, for the case of higher-twist operators. Although calculating the mixing matrix is not very difficult, extraction of the information about eigenvalues of the latter needs a considerable amount of effort. Indeed, if one does not have any idea about the structure of the eigenvectors, the only way to obtain the eigenvalues is to solve a characteristic equation. This, however, is an almost hopeless task. Nonetheless, let us imagine that one has a guess as to the form of an eigenfunction; then there are no problem with evaluating the corresponding eigenvalue. (Note that the exact solutions in [6, 8] and [5], where the analogous problem was investigated for the twist-3 operators in QCD were obtained precisely in this manner.) Thus the more promising strategy is to guess an approximate structure of eigenfunctions in the 'asymptotic' region. However, the simple criterion for determining whether a given vector is close to some eigenvector exists only for Hermitian matrices (see section 2).

Thus for the successful analysis of the asymptotic behaviour of anomalous dimensions two ingredients, namely the hermiticity of the mixing matrix and a proper ansatz of the test vector, are essential. It is not evident that first condition can be satisfied at all. However, for the model under consideration one can choose the scalar product in a such way that a mixing matrix will be Hermitian [6, 7]. Some arguments in favour of it being possible in the general case will be given in section 4. As to the choice of the test vector, this will be discussed below.

The paper is organized as follows: in section 2 we shall introduce notation, derive some formulae and give the exact formulation of the problem; section 3 is devoted to the proof of the theorem about asymptotic behaviour of anomalous dimension, which is the main result of this paper; in the last section we discuss the results obtained.

2. Preliminary remarks

It was shown in [6, 7] that the problem of calculation of anomalous dimensions of the traceless and symmetric composite operators in the scalar ϕ^4 theory in the one-loop approximation is equivalent to the eigenvalue problem for the Hermitian operator H acting on a Fock space $\mathcal{H}$:

$$\mathrm{H} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{i=0}^{n} a_i^\dagger a_{n-i}^\dagger \sum_{j=0}^{n} a_j a_{n-j}. \tag{2.1}$$

Here $a_i^\dagger, a_i$ are the creation and annihilation operators with the standard commutation relations $[a_i, a_k^\dagger] = \delta_{ik}$. The eigenvalues of H and the anomalous dimensions of composite operators are simply related: $\gamma_{an} = \epsilon/3 \cdot \lambda + \mathrm{O}(\epsilon^2)$. There is also a one-to-one correspondence between the eigenvectors of H and the multiplicatively renormalized composite operators [7].

It can easily be shown that H commutes with the N-particle operator and with the $SL(2, C)$ group generators S, S_+, S_-:

$$[\mathrm{S}_-, \mathrm{S}] = \mathrm{S}_- \qquad [\mathrm{S}_+, \mathrm{S}] = -\mathrm{S}_+ \qquad [\mathrm{S}_+, \mathrm{S}_-] = 2\mathrm{S}. \tag{2.2}$$

They can be written as

$$N = \sum_{j=0}^{\infty} a_j^\dagger\, a_j \qquad \mathrm{S} = \sum_{j=0}^{\infty} (j+1/2) \cdot a_j^\dagger\, a_j \qquad \mathrm{S}_- = \sum_{j=0}^{\infty} (j+1) \cdot a_j^\dagger\, a_{j+1} \tag{2.3}$$

and $\mathrm{S}_+ = -\mathrm{S}_-^\dagger$.

Furthermore, due to the commutativity of H with the $SL(2, C)$ generators, each of the subspaces $\mathcal{H}_n^l$ and $\bar{\mathcal{H}}_n^l \in \mathcal{H}_n^l$ $(n, l = 0, \ldots, \infty)$, namely

$$\mathcal{H}_n^l = \{\psi \in \mathcal{H} | N\psi = n\psi\ , \mathrm{S}\psi = (l + n/2)\psi\} \qquad \bar{\mathcal{H}}_n^l = \{\psi \in \mathcal{H}_\backslash^\updownarrow | \mathrm{S}_-\psi = 0\} \tag{2.4}$$

are invariant subspaces of the operator H. Since every eigenvector from $\mathcal{H}_n^l$ which is orthogonal to $\bar{\mathcal{H}}_n^l$ has the form [8]:

$$|\psi\rangle = \sum_k c_k \mathrm{S}_+^k |\psi_\lambda\rangle \qquad |\psi_\lambda\rangle \in \bar{\mathcal{H}}_n^l$$

to obtain all spectrum of the operator H it is sufficient to solve the eigenvalue problem for H on each $\bar{\mathcal{H}}_n^l$ separately.

Moreover, there exists a large subspace of the eigenvectors with zero eigenvalues in each $\bar{\mathcal{H}}_n^l$. They have been completely described in [7] and will not be considered here.

As for non-zero eigenvalues, although at finite l the spectrum of H has a very complicated structure (the numerical results for particular values of n and l are given in [7, 9]), at large l considerable simplifications take place, as will be shown below.

The main result of the present work can be formulated in the form of the following theorem.

Theorem 1. Let the eigenvectors $\psi_1 \in \bar{\mathcal{H}}_n^r$ and $\psi_2 \in \bar{\mathcal{H}}_m^s$ $(\psi_1 \neq \psi_2)$ of the operator H have the eigenvalues λ_1 and λ_2, respectively. Then there exists a number L such that for every $l \geqslant L$ there exists eigenvector $\psi^l \in \bar{\mathcal{H}}_{(m+n)}^l$ with the eigenvalue λ_l such that

$$|\lambda_l - \lambda_1 - \lambda_2| \leqslant C\sqrt{\ln l}/l \tag{2.5}$$

where C is some constant independent of l. In the case where $\psi_1 = \psi_2$ the same inequality holds only for even $l \geqslant L$.

The proof is based on a simple observation. Since any of subspaces $\bar{\mathcal{H}}_n^l$ has a finite dimension, operator H restricted on $\bar{\mathcal{H}}_n^l$ has only point-like spectrum. In this case it can be easily shown that if there is a vector ψ, for which the condition

$$\|(\mathrm{H} - \tilde{\lambda})\psi\| \leqslant \epsilon \|\psi\| \tag{2.6}$$

is fulfilled, then there exists the eigenvector ψ_λ $(\mathrm{H}\psi_\lambda = \lambda\psi_\lambda)$, such that $|\lambda - \tilde{\lambda}| \leqslant \epsilon$. Indeed, expanding a vector ψ in the basis of the eigenvectors of H $\psi = \sum_k c_k \psi_k$ we obtain

$$\epsilon \|\psi\| \geqslant \|(\mathrm{H} - \tilde{\lambda})\psi\| = \left(\sum_k (\lambda_k - \tilde{\lambda})^2 c_k^2\right)^{1/2} \geqslant \min_k |\lambda_k - \tilde{\lambda}| \cdot \|\psi\|.$$

So, to prove the theorem it is sufficient to find out in each subspace $\bar{\mathcal{H}}^l_{n+m}$ a vector which satisfies the corresponding inequality. Note that these arguments are not applicable to a non-Hermitian matrix.

Before proceeding to the proof, we give another formulation of the eigenvalue problem for the operator H. Let us note that there exists a one-to-one correspondence between the vectors from $\mathcal{H}^l_n$ and the symmetric homogeneous polynomials in degree l of n variables:

$$|\Psi\rangle = \sum_{\{j_i\}} c_{j_1,\ldots,j_n} a^\dagger_{j_1}\cdots a^\dagger_{j_n}|0\rangle \to \psi(z_1,\ldots,z_n) = \sum_{\{j_i\}} c_{j_1,\ldots,j_n} z_1^{j_1}\cdots z_n^{j_n} \tag{2.7}$$

the coefficient $c_{j_1,\ldots,j_n}$ being assumed totally symmetric. It is evident that this mapping can be continued to all space.

The operators S, S_+, S_- and H in the n-particle sector take the form [8]:

$$\begin{aligned} \mathrm{S} &= \sum_{i=1}^n (z_i\partial_{z_i} + \tfrac{1}{2}) \qquad \mathrm{S}_- = \sum_{i=1}^n \partial_{z_i} \\ \mathrm{S}_+ &= -\sum_{i=1}^n (z_i^2\partial_{z_i} + z_i) \qquad \mathrm{H} = \sum_{i<k}^n \mathrm{H}(z_i, z_k). \end{aligned} \tag{2.8}$$

The action two-particle Hamiltonian $\mathrm{H}(z_i, z_k)$ on the functions $\psi(z_1,\ldots,z_n)$ reads

$$\mathrm{H}(z_i,z_k)\psi(\ldots,z_i,\ldots,z_k,\ldots) = \int_0^1 \mathrm{d}\alpha\ \psi(\ldots,\alpha z_i + (1-\alpha)z_k,\ldots,\alpha z_i + (1-\alpha)z_k,\ldots). \tag{2.9}$$

It should be stressed that not only H, but also every $\mathrm{H}(z_i, z_k)$ commutes with S, S_+, S_-.

For further calculations it is very convenient to put into correspondence with every function of n variables another one by the following formula [8]:

$$\psi(z) = \sum_{\{j_i\}} c_{j_1,\ldots,j_n} z_1^{j_1}\cdots z_n^{j_n} \quad\to\quad \phi(z) = \sum_{\{j_i\}} (j_1!\cdots j_n!)^{-1} c_{j_1,\ldots,j_n} z_1^{j_1}\cdots z_n^{j_n}. \tag{2.10}$$

The function ψ can be expressed in terms of ϕ in the compact form

$$\psi(z_1,\ldots,z_n) = \phi(\partial_{x_1},\ldots,\partial_{x_n})\prod_{i=1}^n \frac{1}{(1-x_iz_i)}\bigg|_{x_1=\ldots=x_n=0}. \tag{2.11}$$

Then one obtains the following expression for the scalar product for two vectors from $\mathcal{H}^l_n$:

$$\langle\psi_1|\psi_2\rangle_{\mathcal{H}} = n!\cdot\phi_1(\partial_{z_1},\ldots,\partial_{z_n})\psi_2(z_1,\ldots,z_n)|_{z_1=\ldots=z_n=0}. \tag{2.12}$$

It is now easy to check that the operators S, S_+, S_-, H on the space of the 'conjugated' functions $\phi(z_1,\ldots,z_n)$ look like

$$\mathrm{S} = \sum_{i=1}^n (z_i\partial_{z_i} + \tfrac{1}{2}) \qquad \mathrm{S}_+ = \sum_{i=1}^n z_i \qquad \mathrm{S}_- = -\sum_{i=1}^n (z_i\partial^2_{z_i} + \partial_{z_i}) \tag{2.13}$$

$$H\phi(z_1,\ldots,z_n) = \sum_{i<k}\int_0^1 \mathrm{d}\alpha\ \phi(z_1,\ldots,\underset{(i)}{(1-\alpha)(z_i+z_k)},\ldots,\underset{(k)}{\alpha(z_i+z_k)},\ldots,z_n). \tag{2.14}$$

Up to now we have assumed the functions $\psi(z_1,\ldots,z_n)$ to be totally symmetric. However, in what follows we shall deal with non-symmetric functions as well. To treat them on equal footing it is useful to enlarge the region of the definition of the operators S, S_+, S_-, H up to the space of all polynomial functions $\mathcal{B} = \bigoplus_{n,l=0}^\infty \mathcal{B}^l_n$, where $\mathcal{B}^l_n$ is the linear space of the homogeneous polynomials of degree l of n variables with the scalar product given by (2.12). Then the Fock space $\mathcal{H}$ will be isomorphic to the subspace of the symmetric functions of $\mathcal{B}$; the subspace $\bar{\mathcal{H}}_n^l$ will also be isomorphic to the subspace of the symmetric homogeneous translation invariant polynomials of degree l of n variables $\hat{\mathcal{B}}_n^l \in \mathcal{B}_n^l$.

3. Proof of theorem 1

3.1. Part 1

Let us consider two eigenvectors of H: $\psi_1 \in \bar{\mathcal{H}}_n^r$ and $\psi_2 \in \bar{\mathcal{H}}_m^s$ $(\mathrm{H}\psi_{1(2)} = \lambda_{1(2)}\psi_{1(2)})$; and let $\psi_1(x_1, \ldots, x_n)$ and $\psi_2(y_1, \ldots, y_m)$ be the symmetric translation-invariant homogeneous polynomials corresponding to them, of degree r and s respectively. To prove the theorem it is enough to pick out in the subspace $\hat{\mathcal{B}}_{n+m}^l$ (or, the same, in the $\bar{\mathcal{H}}_{n+m}^l$) the function, for which the inequality (2.6) holds.

Let us now consider the following (still non-symmetric) function:

$$\psi^l(x, y) = \sum_{k=0}^{l} c_k(\mathbf{Ad}^k \mathrm{S}_+)\psi_1(x)(\mathbf{Ad}^{l-k}\mathrm{S}_+)\psi_2(y) \tag{3.1.1}$$

where $c_k = (-1)^k C_k^l \cdot C_{k+A}^{l+A+B}$ (C_k^l is the binomial coefficient), $A = n + 2r - 1$, $B = m + 2s - 1$, $\mathbf{Ad}\mathrm{S}_+ = [\mathrm{S}_+, \cdot]$, and where for brevity we have used the notation $\psi^l(x, y)$ for $\psi^l(x_1, \ldots, x_n, y_1, \ldots, y_m)$, and $\psi_{1(2)}(x)$ for $\psi_{1(2)}(x_1, \ldots, x_{n(m)})$.

Using equation (2.2) it is easy to check that the function ψ given by (3.1.1) is translation invariant, i.e. $\mathrm{S}_-\psi^l(x, y) = 0$.

To construct the test function we symmetrize $\psi^l(x, y)$ over all x and y:

$$\psi_S^l(x, y) = Sym_{\{x,y\}}\psi^l(x, y) = \frac{n!m!}{(n+m)!}\sum_{k=0}^{m}\sum_{\substack{\{i_1<\cdots<i_k\}\\ \{j_1<\cdots<j_k\}}}\psi_{(j_1\ldots j_k)}^{(i_1\ldots i_k)}(x, y) \tag{3.1.2}$$

where $\psi_{(j_1\ldots j_k)}^{(i_1\ldots i_k)}(x, y)$ is obtained from $\psi^l(x, y)$ by interchanging $x_{i_1} \leftrightarrow y_{j_1}$, and so on. Also, without loss of generality, hereafter we take $n \geqslant m$. (For the cases $m = 1$ and $n = 1(2)$ the expression for the $\psi_S^l(x, y)$ yields the exact eigenfunctions, so one might hope that it will also be a good approximation in other cases.)

The corresponding expression for the 'conjugate' function $\phi^l(x, y)$ looks more simple. Really, taking into account that $\mathrm{S}_+\phi(\cdots) = (x_1 + \cdots + x_n)\phi(\cdots)$ one obtains

$$\phi^l(x, y) = \phi_1(x)\phi_2(y)K(a, b)\exp a\left(\sum_{i=1}^{n} x_i\right)\exp b\left(\sum_{j=1}^{m} y_j\right)\Bigg|_{a=b=0} \tag{3.1.3}$$

where

$$K(a, b) \equiv \sum_{k=0}^{l} c_k \partial_a^k \partial_b^{(l-k)}. \tag{3.1.4}$$

Then, with the help of equations (2.11), (3.1.3) the following representation for the function $\psi^l(x, y)$ can be derived:

$$\psi^l(x, y) = \phi_1(\partial_\xi)\phi_2(\partial_\eta)K(a, b)\prod_{i,j}\frac{1}{(1 - x_i(a + \xi_i))}\frac{1}{(1 - y_i(b + \eta_i))}\Bigg|_{(a,b,\xi_i,\eta_j)=0}. \tag{3.1.5}$$

Now we will show that the inequality (2.6) with $\lambda = \lambda_1 + \lambda_2$ and $\epsilon = C\sqrt{\ln l}/l$ holds for the function $\psi_S^l(x, y)$. As was mentioned before, this is sufficient to the prove the theorem.

Our first task is to obtain the estimate from below of the norm of the function $\psi_S^l(x,y)$ for large values of l. Using equations (2.12), (3.1.5) and taking into account the fact that $\psi_S^p(x,y)$ is totally symmetric, one gets

$$\|\psi_S^l\|^2 = n!m!\sum_{k=0}^{m}\sum_{\substack{\{i_1<\cdots<i_k\}\\ \{j_1<\cdots<j_k\}}}\phi^l(\partial_x,\partial_y)\psi^{(i_1\ldots i_k)}_{(j_1\ldots j_k)}(x,y) = n!m!\sum_{k=0}^{m}C_k^nC_k^mA_l^{(k)}. \tag{3.1.6}$$

The coefficients $A_l^{(k)}$ are given by the equation

$$A_l^{(k)} = \mathcal{N}\phi^l(\partial_x,\partial_y)\psi^{(1\ldots k)}_{(1\ldots k)}(x,y), = \mathcal{N}\phi_1(\partial_{x_i})\phi_2(\partial_{y_i})K(a,b)\psi^{(1\ldots k)}_{(1\ldots k)}(x,y+b-a) \tag{3.1.7}$$

where the symbol $\mathcal{N}$ means that all arguments must be set to zero at the end of calculation. The substitution of (3.1.3), (3.1.5) in (3.1.7) yields

$$A_l^{(k)} = \mathcal{N}\phi_1(\partial_x)\phi_2(\partial_y)\phi_1(\partial_{\bar x})\phi_2(\partial_{\bar y})K(a,b)K(\bar a,\bar b)\left[\prod_{i=k+1}^{m}\frac{1}{(1-(\bar y_i+\bar b-\bar a)(y_i+b))}\right.$$
$$\left.\times\prod_{i=1}^{k}\frac{1}{(1-(\bar y_i+\bar b-\bar a)(x_i+a))(1-\bar x_i(y_i+b))}\prod_{i=k+1}^{n}\frac{1}{(1-\bar x_i(x_i+b))}\right]. \tag{3.1.8}$$

The expression in the square brackets in (3.1.8) depends only on the difference $\bar b-\bar a$; hence

$$K(\bar a,\bar b)\Big[\cdots\Big]_{\bar a,\bar b\ldots=0} = \left(\sum_k(-1)^kc_k\right)\partial_{\bar b}^l\Big[\cdots\Big]_{\bar a,\bar b\ldots=0}.$$

In the resulting expression the dependence on $\bar b$ can be factorized after the appropriate rescaling of the variables $x,y,\bar x,\bar y,a,b$. Finally, taking advantage of (2.11) and remembering that the function ψ_1, ψ_2 (but not $\phi(\cdots)$) are translation invariant, one gets

$$A_l^{(k)} = Z\mathcal{N}\phi_1(\partial_{x_i})\phi_2(\partial_{y_i})K(a,b)F_k(a,b,x,y) \tag{3.1.9}$$

where $Z = l!\sum_{k=0}^{l}c_k(-1)^k = l!C_{l+A}^{2l+A+B}$ and

$$F_k(a,b,x,y) = \left[\psi_1(y_1,\ldots,y_k,x_{k+1}+a-b,\ldots,x_n+a-b)\prod_{i=1}^{k}\frac{1}{1-x_i-a}\right.$$
$$\times\prod_{i=k+1}^{m}\frac{1}{1-y_i-b}(1-a)^{-s}\psi_2$$
$$\left.\times\left(\frac{x_1}{1-x_1-a},\ldots,\frac{x_k}{1-x_k-a},\frac{y_{k+1}+b-a}{1-y_{k+1}-b},\ldots,\frac{y_m+b-a}{1-y_m-b}\right)\right]. \tag{3.1.10}$$

It is easy to understand that after the differentiation with respect to x_i, y_j the resulting expression will have the form

$$A_l^{(k)} = Z\mathcal{N}K(a,b)\sum_{n_1,n_2,n_3}C^k_{n_1,n_2,n_3}\frac{(a-b)^{n_1}}{(1-a)^{n_2}(1-b)^{n_3}} = Z\sum_{n_1,n_2,n_3}C^k_{n_1,n_2,n_3}A^k_{n_1,n_2,n_3}(l) \tag{3.1.11}$$

the summation over $n_1,\ n_2,\ n_3$ being carried out in the limits, which as well as the coefficients $C^k_{n_1,n_2,n_3}$ are independent of the parameter l. Thus, all dependence on l of $A_l^{(k)}$, except for the trivial factor Z, is contained in the coefficients $A^k_{n_1,n_2,n_3}$.

Our further strategy is as follows. First of all, we shall obtain the result for the quantity $A_l^{(0)}$ and for $A_l^{(m)}$. (These terms gives the main contributions to the norm of the vector ψ^l_S.) Then, we shall show (it will be done in the appendix) that for all other $A_l^{(k)}$ $(1 \leqslant k \leqslant m-1)$ the ratio $A_l^{(k)}/A_l^{(0)}$ tends to zero as $1/l^2$ at least.

To calculate $A_l^{(0)}$ it is sufficient to note that after the appropriate shift of the arguments in the functions ψ_1 and ψ_2 the expression for $F_0(a,b,x,y)$ (equation (3.1.10)) reads

$$\left[\psi_1(x_1,\ldots,x_n)\prod_{i=1}^m \frac{1}{1-y_i-b}(1-b)^{-s}\psi_2\left(\frac{y_1}{1-y_1-b},\ldots,\frac{y_m}{1-y_m-b}\right)\right]. \tag{3.1.12}$$

Then carrying out the differentiation with respect to $x_i,\ y_j$ in (3.1.9) one obtains

$$A_l^{(0)} = Z(m!n!)^{-1}\|\psi_1\|^2\|\psi_2\|^2\mathcal{N}K(a,b)(1-b)^{-(2s+m)} = (m!n!)^{-1}\mathcal{A}(l) \tag{3.1.13}$$

where $\mathcal{A}(l) = C^{2l+A+B}_{l+A}\, l!(l+A+B)!\|\psi_1\|^2\|\psi_2\|^2/(A!B!)$.

The evaluation of A_l^m in the case where $n=m=k$ differs from that considered above only in the interchange of variables $x,a \leftrightarrow y,b$ in (3.1.12), this resulting in

$$A_l^{(m)} = Z|\langle\psi_1|\psi_2\rangle|^2\mathcal{N}K(a,b)(1-a)^{-(2s+m)} = (-1)^l A_l^{(0)}\delta_{\psi_1\psi_2}. \tag{3.1.14}$$

Here, we take into account that $\psi_1,\ \psi_2$ are the eigenfunctions of the self-adjoint operator. Thus, when l is odd and $\psi_1 = \psi_2$ these two contributions ($A^{(0)}$ and $A^{(m)}$) cancel each other. However, as one can easily see from (3.1.1), the function $\psi^l(x,y)$ is identically equal to zero in this case.

In the case where $k=m$ and $m<n$, the expression for $A_l^{(m)}$ (equation (3.1.11)) takes the form $A_l^{(m)} = Z\mathcal{N}K(a,b)\sum_{n_1=s}^r C^{(m)}_{n_1}(a-b)^{n_1-s}(1-a)^{-(s+n_1+)}$. After some algebra one obtains

$$A_l^{(m)} = \mathcal{A}(l)\sum_{k=0}^{r-s}\sum_{i=0}^{k}\tilde{C}_{k,i}C^l_i C^{l+B+k-i}_{l+A-i} \leqslant C\cdot A_l^{(0)}/l^{n-m} \leqslant CA_l^{(0)}/l. \tag{3.1.15}$$

For all other $A_l^{(k)}$ $(0<k<m)$ we are able to derive the following estimate (see the appendix for details): $|A_l^{(k)}| \leqslant C_k A_l^{(0)}/l^2$, (here the C_k are some constants). Then, taking into account this result together with equations (3.1.13), (3.1.14) (3.1.15), one obtains

$$\|\psi^l_S\|^2 = (1+(-1)^l\delta_{\psi_1,\psi_2})\mathcal{A}(l)(1+\mathrm{O}(1/l)). \tag{3.1.16}$$

3.2. Part 2

To complete the proof one should obtain the following inequality for large l:

$$\epsilon(l) = \|\delta\mathrm{H}\psi^l_S\|^2 = \|(\mathrm{H}-\lambda_1-\lambda_2)\psi^l_S\|^2 \leqslant C\ln l/l^2\mathcal{A}(l). \tag{3.2.1}$$

First of all, let us show that $\epsilon(l)(l)$ can be estimated as follows:

$$\epsilon(l) \leqslant (mn)^2\langle\psi^l(x,y)H(x_1,y_1)\psi^l(x,y)\rangle. \tag{3.2.2}$$

Indeed, using representation (3.1.2) for ψ^l_S in (3.2.1) and taking into account the invariance of the operator H under any transposition of its arguments (see equation (2.8)) one gets

$$(\epsilon(l))^{1/2} = \frac{n!m!}{(n+m)!}\left\|\sum_{k=0}^m\sum_{\substack{\{i_1<\cdots<i_k\}\\ \{j_1<\cdots<j_k\}}}\delta\mathrm{H}\psi^{(i_1\ldots i_k)}_{(j_1\ldots j_k)}(x,y)\right\| \leqslant \|\delta\mathrm{H}\psi^l(x,y)\|. \tag{3.2.3}$$

Moreover, from equations (2.8), (3.1.1) the equality $\delta \mathrm{H}\psi^l(x,y) = \sum_{i,k} H(x_i,y_k)\psi^l(x,y)$ immediately follows. Then to obtain (3.2.2) it is sufficient to note that

$$\|H(x_i,y_k)\psi^l(x,y)\|^2 = \|H(x_1,y_1)\psi^l(x,y)\|^2 = \langle\psi^l(x,y)H(x_1,y_1)\psi^l(x,y)\rangle.$$

In turn, with the help of equations (3.1.3), (3.1.5), the matrix element in (3.2.2) can be represented in a form similar to that for A_l^k (equation (3.1.11)):

$$\langle\psi^l(x,y)H(x_1,y_1)\psi^l(x,y)\rangle = Z(n+m)!K(a,b)\phi_1(\partial_x)\phi_2(\partial_y)\prod_{i=2}^n \frac{1}{1-ax_i}\prod_{i=2}^m\frac{1}{1-by_i}$$
$$\times\int_0^1 \mathrm{d}s\,\frac{1}{1-a\theta(s)}\frac{1}{1-b\theta(s)}\psi_1\left(\frac{\theta(s)}{1-a\theta(s)},\frac{1}{1-ax_i}\right)$$
$$\times\psi_2\left(\frac{\theta(s)}{1-b\theta(s)},\frac{1}{1-by_i}\right)\Bigg|_{\substack{x=0,y=1\\a=b=0}} \tag{3.2.4}$$

where $\theta(s) = sx_1+(1-s)y_1$. Differentiating with respect to $x_i,\ y_j, i,j>1$ one gets

$$\langle\cdots\rangle = \sum_{n_1=0}^r\sum_{n_2=n_1}^r\sum_{m_1=0}^s\sum_{m_2=0}^s\sum_{m_3=0}^{s-m_1} c^{n_1n_2}_{m_1,m_2,m_3}\tilde{a}^{n_1n_2}_{m_1,m_2,m_3}(l) \tag{3.2.5}$$

where the coefficients $c^{n_1n_2}_{m_1,m_2,m_3}$ do not depend on l, but

$$\tilde{a}^{n_1n_2}_{m_1,m_2,m_3}(l) = ZK(a,b)\left[\frac{a^{n_2-n_1}}{(1-b)^{\beta-m_2}}\partial_x^{n_1}\partial_y^{m_1}\int_0^1\mathrm{d}s\,\frac{\theta^{n_2+m_2}}{(1-a\theta)^{n_2+1}(1-b\theta)^{m_2+1}}\right]_{\substack{x=0,y=1\\a=b=0}} \tag{3.2.6}$$

and $\beta = s+m_3+m-1$.

Before applying the operator $K(a,b) = \sum_{k=0}^l c_k\partial_a^k\partial_b^{l-k}$ to the expression in the square brackets it is convenient to rewrite the latter in a form more suitable for this purpose:

$$\Big[\cdots\Big] = (n_2!m_2!)^{-1}\frac{a^{n_2-n_1}}{(1-b)^{\beta-m_2}}\int_0^1\mathrm{d}s\,s^{m_1}(1-s)^{n_1}\partial_a^{n_2}\partial_b^{m_2}\partial_s^{(m_1+n_1)}\frac{1}{(1-as)(1-bs)}. \tag{3.2.7}$$

Now all differentiations with respect to a and b can be carried out easily:

$$\partial_a^k a^{n_2-n_1}\partial_a^{n_2}\,\frac{1}{(1-as)}\Bigg|_{a=0} = (k+n_1)!s^{k+n_1}\partial_x^{n_2-n_1}x^k\Big|_{x=1}$$

$$\partial_b^{l-k}\frac{1}{(1-b)^\beta}\partial_b^{m_2}\frac{1}{(1-sb)} = s^{m_2}\frac{(l-k+\beta)!}{\Gamma(\beta-m_2)}\int_0^1\mathrm{d}\alpha\,\alpha^{\beta-m_2-1}(1-\alpha)^{m_2}[\alpha+(1-\alpha)s]^{l-k}.$$

Finally, after a representation of the ratio of the factorials like $(k+n_1)!/(k+A)!$ in the form $1/\Gamma(A-n_1) = \int_0^1\mathrm{d}u\,u^{k+n_1}(1-u)^{A-n_1-1}$, the summation over k becomes trivial and we obtain the following expression for $\tilde{a}^{n_1n_2}_{m_1,m_2,m_3}(l)$:

$$\tilde{a}^{n_1n_2}_{m_1,m_2,m_3}(l) = \mathcal{A}(l)a^{n_1n_2}_{m_1,m_2,m_3}(l) \tag{3.2.8}$$

where

$$a^{n_1n_2}_{m_1,m_2,m_3}(l) = \frac{1}{\Gamma(A-n_1)\Gamma(\beta-m_2)\Gamma(B-\beta)}\partial_x^{n_2-n_1}$$
$$\times\int_0^1\cdots\int_0^1\mathrm{d}s\,\mathrm{d}\alpha\,\mathrm{d}u\,\mathrm{d}v\,u^{n_1}(1-u)^{A-n_1-1}s^{m_1}(1-s)^{n_1}\partial_s^{n_1+m_1}s^{n_1+m_2}v^\beta$$

$$\times(1-v)^{B-\beta-1}\alpha^{\beta-m_2-1}(1-\alpha)^{m_2}[v(\alpha+(1-\alpha)s)-sux]^l\Big|_{x=1}. \tag{3.2.9}$$

Note that when the arguments of the Γ functions become equal to zero, the following evident changes must be made: $1/\Gamma(A-n_1)\int \mathrm{d}u\,(1-u)^{A-n_1-1}\cdots \to \int \mathrm{d}u\,\delta(1-u)$ if $A=n_1$, and so on. Taking account of (3.2.8), equation (3.2.5) now reads

$$\langle\psi^l(x,y)H(x_1,y_1)\psi^l(x,y)\rangle = \mathcal{A}(l)\sum_{n_1=0}^{r}\sum_{n_2=n_1}^{r}\sum_{m_1=0}^{s}\sum_{m_2=0}^{s}\sum_{m_3=0}^{s-m_1} c^{n_1n_2}_{m_1,m_2,m_3}a^{n_1n_2}_{m_1,m_2,m_3}(l). \tag{3.2.10}$$

Thus to prove the inequality (3.2.1) for $\epsilon(l)$ we need only show that the coefficients $a^{n_1n_2}_{m_1,m_2,m_3}(l)$ tend to zero at $l\to\infty$ no more slowly than $\ln l/l^2$.

First of all, let us consider the cases where at least one Γ function in (3.2.9) has a zero argument. (It is worth recalling that $A=2r+n-1$, $B=2s+m-1$; $n,\ m$ $(m\leqslant n)$ are the numbers of variables and r,s are respectively the degrees of the translation invariant polynomials ψ_1 and ψ_2).

1. $A=n_1$. It is evident that this equality is possible only when $n=1,\ r=0$, and hence $m=1,\ s=0$. Then one immediately obtains the result that the arguments of two other Γ functions are also zero, and the corresponding integral (see equation (3.2.9) and the note to it) is zero.

2. $\beta-m_2=0,\ n>1$ $(0<A-n_1)$. In this case one obtains $m=1,\ s=0$ and $B-\beta=0$. Since two of the Γ functions have arguments equal to zero, the integrations over v and α are removed. After this it is trivial to check that $a(l)$ tends to zero as $1/l^2$ at $l\to\infty$.

3. $B-\beta=0$ and $1<m\leqslant n$. These conditions imply that $m_1=0$ and $m_3=s$. Again the integration over v is removed. To calculate $a(l)$, let us write $\partial_x^{n_2-n_1}$ in (3.2.9) as $u^{n_2-n_1}\partial_u^{n_2-n_1}$ and carry out the integration by parts as over u as well as over s. Note that the boundary terms do not appear when $m_1=0$. Then it is clear that integrand represents the product of two functions, one of which, $[(\alpha+(1-\alpha)s)-su]^l$, is positive definite in the region of integration and the other is the sum of the monomials like $s^{i_1}(1-s)^{i_2}\alpha^{i_2}\cdots$ with finite coefficients and, hence, can be limited by some constant independent of l. Then, taking into account this remark, one obtains the following estimate for $a(l)$:

$$|a^{n_1n_2}_{m_1,m_2,m_3}(l)|\leqslant C\int_0^1 \mathrm{d}s\ \mathrm{d}\alpha\ \mathrm{d}u\ [(\alpha+(1-\alpha)s)-su]^l = \frac{2C[\psi(l+2)-\psi(1)]}{(l+1)(l+2)} \tag{3.2.11}$$

where $\psi(x)=\partial_x\ln\Gamma(x)$.

4. Finally, we consider the case when all arguments of Γ functions in (3.2.9) are greater then zero. As in the previous case it is convenient to replace $\partial_x^{n_2-n_1}$ by $u^{n_2-n_1}\partial_u^{n_2-n_1}$ and fulfil the integration over u and s by parts. However, the boundary terms arise now with the integration over s at upper bound $(s=1)$. However, it is not hard to show that each of them decreases as $1/l^2$ at $l\to\infty$. (All calculations practically repeat those given in the appendix.)

The last term to be calculated has the form

$$I(l)=\iiint \mathrm{d}s\ \mathrm{d}\alpha\ \mathrm{d}u\ \mathrm{d}v\ A(s,\alpha,u,v)[v(\alpha+(1-\alpha)s)-su]^l \tag{3.2.12}$$

where $A(s,\alpha,u,v)$ is some polynomial of the variables s,α,u,v, such that $|A(s,\alpha,u,v)|<C$ in the domain $0\leqslant s,\ \alpha,\ u,\ v\leqslant 1$.

To derive the necessary estimate in the case of even l it is sufficient to replace $A(s, \alpha, u, v)$ in (3.2.12) by a constant C. For odd l, integral under consideration can be estimated from above in the following way:

$$I(l) \leqslant C\left(\int_{\Omega_+} - \int_{\Omega_-}\right) \mathrm{d}s\ \mathrm{d}\alpha\ \mathrm{d}u\ \mathrm{d}v\ [v(\alpha + (1-\alpha)s) - su]^l$$

where the regions Ω_+ and Ω_- are determined from conditions: $[v(\alpha + (1-\alpha)s) - su] > 0$ or < 0, respectively. The evaluation of the corresponding integrals does not cause any trouble and leads to the following result:

$$I(l) \leqslant C \ln l / l^2. \tag{3.2.13}$$

Then, taking into account equations (3.2.1), (3.2.2), (3.2.10) and (3.1.16), one concludes that there exist such constants L and C that the inequality

$$\|(\mathrm{H} - \lambda_1 - \lambda_2)\psi_S^l\|^2 \leqslant C \ln l / l^2 \|\psi_S^l\|^2 \tag{3.2.14}$$

holds for all $l \geqslant L$. This inequality, as shown in section 2, guarantees the existence of the eigenvector of the operator H with the eigenvalue satisfying (2.5). □

4. Conclusions

The theorem proved in the previous section provides a number of consequences for the spectrum of the operator H:

- Every point of the spectrum is either a limit point of the latter or an exact eigenvalue of infinite multiplicity.
- Any finite sum of eigenvalues and limit points of the spectrum is a limit point again.

These statements follow directly from the theorem.

Furthermore, let us denote by $\mathcal{S}_n$ the spectrum of the operator H restricted on n-particle sector of Fock space, $(N\psi = n\psi)$ and by $\bar{\mathcal{S}}_n$ the set of the limit points of $\mathcal{S}_n$. Then it is easy to see that the definite relations ('hierarchical structures') between $\mathcal{S}_n$, $\bar{\mathcal{S}}_n$ $(n = 2, \ldots \infty)$ exist. Indeed, let Σ_n is the set of all possible sums of $s_{i_1} + s_{i_2} + \cdots + s_{i_m}$ type, where $s_{i_k} \in \mathcal{S}_k$ and $i_1 + \cdots + i_m \leqslant n$, $i_1 \leqslant i_2 \leqslant n \cdots \leqslant i_m$. Then one can easily conclude that $\Sigma_n \subset \bar{\mathcal{S}}_n$. For $n = 2, 3$ the stronger equations $\Sigma_n = \bar{\mathcal{S}}_n$ hold, but the examination of this conjecture for a general n requires additional analysis.

It should also be noted that the structure of the spectrum bears some resemblance (this may be purely speculative) to those in 2D conformal field theories. The well known peculiarity of 2D CFT is that all admissible states can be divided into definite classes, namely Verma modules. The dimensions of state (the critical exponents of the corresponding observables) in each Verma module differ one from another by an integer. Moreover, there exists a large class of so-called minimal models, in which the number of Verma modules (or, the same, primary fields) is finite [4]. Let us now look at the spectrum of critical exponents in ϕ^4 theory. Although the 'naive addition law' of the anomalous dimensions can hardly be combined with the finiteness of the spectrum, the structures having some similarity with the Verma module are easily noticed. Indeed, one can conclude from the theorem that there exist groups of operators whose full scaling dimensions differing one from another by an 'almost integer number': $\Delta_l - \Delta_{l'} = l - l' + \mathrm{O}(1/\min(l, l'))$.

Below we give some evidence in favour of the claim that the results obtained are common to general D-dimensional CFT, and not specific only to the ϕ^4 model. In this case it will be very interesting to elucidate what changes in the structure of spectrum happen at $D \to 2$.

Moreover, there exists the problem with high-gradient composite operators in $2+\epsilon$ expansion, where the technique developed here might appear to be useful. It has been noticed that in various models (the Q-matrix model [11], N-vector model [12], unitary matrix model [13] and orthogonal matrix model [14]) a certain class of canonically irrelevant composite operators with $2l$ gradients acquires anomalous dimensions to first order in $\epsilon(l)$ that grow with s^2 and thus endangers the stability of the non-trivial fixed points. (The two-loop calculations carried out in [15] make the stability problem even worse.)

Finally, we would like to discuss two question:

1. What properties of the one-loop spectrum of anomalous dimensions in the ϕ^4 model survive to a higher order of the perturbation theory?
2. To what extent are they conditioned by the peculiarity of the ϕ^4 theory?

As to the first question we can only adduce some arguments in favour of a non-perturbative character of the obtained results. In [9] the spectrum of the critical exponents of the N-vector model in $4-\epsilon$ dimensions was investigated to second order in ϵ. In this work it was shown that some one-loop properties of the spectrum, in particular a generic class of degeneracies [7, 8], are lifted to two-loop order. However, the results of the numerical analysis of critical exponents carried out for the operators with number of fields $\leqslant 4$ distinctly show that a limit-point structure of the spectrum is preserved.

The other evidence in favour of this hypothesis can be found in the works of Lang and Rühl [16]. They investigated the spectrum of the critical exponents in the nonlinear σ-model in $2<d<4$ dimensions to first order in the $1/N$ expansion. The results for various classes of composite operators [16] display the existence of a similar limit-point structure in this model for the whole range $2<d<4$. Since the critical exponents should be consistent in the $1/N$ expansion for the σ-model and the $4-\epsilon$ expansion for the $(\phi^2)^2$ model, one may expect this property of the spectrum to hold to all orders in the ϵ.

To answer the second question it is useful to understand what features of the model under consideration determine the properties of the operator H (hermiticity, invariance under $SL(2,C)$ group, two-particle type of interaction) were crucial to the proof of the theorem. The first two properties are closely related to the conformal invariance of the ϕ^4 model [17]. It can be shown that a two-particle form of the operator H and the conformal invariance of a theory lead to hermiticity of H in the scalar product given by 2.12. (The relation between the functions ψ and ϕ in the general case is given in [18].) Furthermore, it should be emphasized that the commutativity of H with S and S_+ reflects two simple facts: (i) Non-trivial mixing occurs (in ϕ^4 theory) only between operators with an equal number of fields. (ii) The total derivative of a eigenoperator is an eigenoperator with the same anomalous dimension as well. However, if the operator H is Hermitian it must commute with the operator conjugated to S_+ as well. So the minimal group of invariance of H ($SL(2,C)$ in our case) has three generators.

One can see that hermicity and $SL(2,C)$ invariance of the operator H follow directly from the conformal invariance of the ϕ^4 model. Thus, the method of analysis of anomalous dimensions presented here is not peculiar to ϕ^4 theory, but can be applied to other conformal-invariant field theories.

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Appendix

In this appendix we calculate the quantities $A_l^{(k)}$ for $1 \leqslant k \leqslant m-1$. Let us recall the representation for $A_l^{(k)}$ (see equation (3.1.11)):

$$A_l^{(k)} = Z\mathcal{N}K(a,b)\sum_{n_1,n_2,n_3} C^k_{n_1,n_2,n_3}\frac{(a-b)^{n_1}}{(1-a)^{n_2}(1-b)^{n_3}} = Z\sum_{n_1,n_2,n_3} C^k_{n_1,n_2,n_3}A^{(k)}_{n_1,n_2,n_3}.$$

The summation over n_1, n_2, n_3 is carried out in the following range:

$$n_1 = m_1 + m_2 \qquad n_2 = k + s + m_1 + m_3 \qquad n_3 = s + m - k + m_2 - m_3$$

$$0 \leqslant m_1 \leqslant \min[s, r] \qquad 0 \leqslant m_2 \leqslant s \qquad 0 \leqslant m_3 \leqslant r - m_1.$$

The simpler way of obtaining these bounds from (3.1.10) is to treat $(a-b)$, $(1-a)$, $(1-b)$ as independent variables.

Then, taking advantage of the Feynman's formula for the product $(1-a)^{-n_2}(1-b)^{-n_3}$ (starting from here we shall omit all inessential multipliers that are independent of l), one obtains

$$A^{(k)}_{n_1,n_2,n_3} \sim \sum_{k=0}^{l} c_k \partial_a^k \partial_b^{(l-k)} \int_0^1 \mathrm{d}x\; x^{n_2-1}(1-x)^{n_3-1}\partial_x^{n_1}[1 - ax - (1-x)b]^{-B-1}. \tag{A.1}$$

Now, for a time, we assume that $r \leqslant s$. Then to obtain the final expression for $A^{(k)}_{n_1,n_2,n_3}$ one needs to carry out the integration by parts in (A.1) (note that there are no boundary terms in the case $r \leqslant s$) and, using the integral representation for the ratio of the factorials arising from the differentiation with respect to a, b, x, carry out the summation over k:

$$A^{(k)}_{n_1,n_2,n_3} \sim \sum_{j=n_3-n_1-1}^{B-1} \alpha_j \frac{(l+B)!(l+B+A)!}{(l+j+1)!}\int_0^1 \mathrm{d}u\;\mathrm{d}v\; u^{A-1}(1-v)^j v^{B-j-1}(u-v)^l. \tag{A.2}$$

Here, the α_j are some inessential constants. It is evident that the leading contributions to the integrals in (A.2) come from regions where $|(u-v)| \simeq 1$ ($u \simeq 1$, $v \simeq 0$ and $u \simeq 0$, $v \simeq 1$) and tend to zero at $l \to \infty$ as l^{-A-1} and l^{-B+j-1}, respectively. Then, taking into account that $s - r + m - k - 1 \leqslant j \leqslant B - 1$ and collecting all necessary terms, one obtains the result that the contribution from $A^{(k)}_{n_1,n_2,n_3}$ to the $A_l^{(k)}$ is of order $\mathcal{A}(l)/l^2$. Thus we have obtained the required result for the case $r \leqslant s$.

To derive the estimate for $A_l^{(k)}$ in the case $s \leqslant r$ in the manner described above, we change the basic formula for $A_l^{(k)}$—eq.(3.1.7) slightly. Using the property of translation invariance of function $\psi^l(x,y)$, one can rewrite the right-hand side of (3.1.7) in the following way: $\mathcal{N}\phi_1(\partial_{x_i})\phi_2(\partial_{y_i})K(a,b)\psi^{(1\ldots k)}_{(1\ldots k)}(x+a-b, y)$. Then the further calculations simply repeat those for the case $r \leqslant s$ and lead to the same estimate for $A_l^{(k)}$.

Thus, we show that the inequality

$$|A_l^{(k)}| \leqslant \text{constant} \times A_l^{(0)}/l^2$$

holds for all $1 \leqslant k \leqslant m-1$.

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